OD-Characterization of Some Linear Groups Over Binary Field and Their Automorphism Groups*

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Abstract

The Gruenberg-Kegel graph $\mathrm{GK}(G) = (V_G, E_G)$ of a finite group G is a simple graph with vertex set $V_G = \pi(G)$, the set of all primes dividing the order of G, and such that two distinct vertices p and q are joined by an edge, $\{p, q\} \in E_G$, if G contains an element of order pq. The degree $\deg_G(p)$ of a vertex $p \in V_G$ is the number of edges incident on p. In the case when $\pi(G) = \{p_1, p_2, \ldots, p_h\}$ with $p_1 < p_2 < \cdots < p_h$, we consider the h-tuple $D(G) = (\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_h))$, which is called the degree pattern of G. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying condition $(|H|, D(H)) = (|G|, D(G))$. Especially, a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we first find the degree pattern of the projevtive special linear groups over binary field $L_n(2)$ and among other results we prove that the simple groups $L_{10}(2)$ and $L_{11}(2)$ are OD-characterizable (Theorem 1.2). It is also shown that automorphism groups $\mathrm{Aut}(L_p(2))$ and $\mathrm{Aut}(L_{p+1}(2))$, where $2^p - 1$ is a Mersenne prime, are OD-characterizable (Theorem 1.3).

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1 Introduction

Throughout this paper, all groups considered are *finite* and simple groups are *non-abelian.* Given a group G, denote by $\pi_e(G)$ the set of order of all elements in G. It is clear that the set $\pi_e(G)$ is *closed* and *partially ordered* by divisibility, hence, it is uniquely determined by $\mu(G)$, the subset of its maximal elements. We also denote by $\pi(n)$ the set of all prime divisors of a positive integer n. For a finite group G, we shall write $\pi(G)$ instead of $\pi(|G|)$.

To every finite group G we associate a graph known as *Gruenberg-Kegel graph* (or *prime graph*) denoted by $\mathrm{GK}(G) = (V_G, E_G)$. For this graph we have $V_G = \pi(G)$, and for two distinct vertices $p, q \in V_G$ we have $\{p, q\} \in E_G$ if and only if $pq \in \pi_e(G)$. When p and q are adjacent vertices in $\mathrm{GK}(G)$ we will write $p \sim q$. Denote the connected components of $\mathrm{GK}(G)$ by $\mathrm{GK}_i(G) = (\pi_i(G), E_i(G))$, $i = 1, 2, \ldots, s(G)$, where $s(G)$ is the number of connected components of $\mathrm{GK}(G)$. If $2 \in \pi(G)$, then we set $2 \in \pi_1(G)$. In the papers [16] and [32] the connected components of the Gruenberg-Kegel graph of all non-abelian finite simple groups are determined. An corrected list of these groups can be found in [17].

Recall that a complete graph is a graph in which every pair of vertices is adjacent. It is worth noting that if S is a simple group with disconnected prime graph, then all connected components $\mathrm{GK}_i(S)$ for $2 \leqslant i \leqslant s(S)$ are complete graphs, for instance, see [28].

When the group G has connected components $\mathrm{GK}_1(G), \mathrm{GK}_2(G), \ldots, \mathrm{GK}_{s(G)}(G)$, $|G|$ can be expressed as the product of $m_1, m_2, \ldots, m_{s(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i(G)$. We call $m_1, m_2, \ldots, m_{s(G)}$ the *order components* of G and we write

$$\mathrm{OC}(G) := \{m_1, m_2, \ldots, m_{s(G)}\},$$

the set of all order components of G.

The *degree* $\deg_G(p)$ *of a vertex* $p \in \pi(G)$ is the number of edges incident on p. When there is no ambiguity on the group G, we denote $\deg_G(p)$ simply by $\deg(p)$. If $\pi(G)$ consists of the primes $p_1, p_2, \ldots, p_h$ with $p_1 < p_2 < \cdots < p_h$, then we define

$$\mathrm{D}(G) := \big(\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_h)\big),$$

which is called the *degree pattern of* G. Moreover, we set

$$\Omega_n(G) := \{p \in \pi(G) | \deg_G(p) = n\},$$

for $n = 0, 1, 2, \ldots, h-1$. Clearly,

$$\pi(G) = \bigcup_{n=0}^{h-1} \Omega_n(G).$$

Moreover, since $\deg_G(p) = 0$ if and only if $(\{p\}, \emptyset)$ is a connected component of $\mathrm{GK}(G)$, we have $|\Omega_0(G)| \leqslant s(G) \leqslant 6$ (see [32]). A group G is called a $C_{p,p}$-group if $p \in \Omega_0(G)$.

Given a finite group M, denote by $h_{\mathrm{OD}}(M)$ the number of isomorphism classes of finite groups G such that $|G| = |M|$ and $\mathrm{D}(G) = \mathrm{D}(M)$. In terms of the function h_{OD}, we have the following definition.

Definition 1.1 *A finite group M is called* k-fold OD-characterizabale *if* $h_{\rm OD}(M) = k$. *Usually, a* 1*-fold OD-characterizabale group is simply called* OD-characterizabale.

The notion of OD-characterizability of a finite group was first introduced by the first author and his colleagues in [26]. It is well-known that, according to Cayley's theorem, for each positive integer n there are only *finitely* many non-isomorphic groups of order n normally denoted by $\nu(n)$. Hence

$$1 \leqslant h_{\rm OD}(G) \leqslant \nu(|G|),$$

for every finite group G, and the following result follows immediately.

Theorem 1.1 *Every finite group is k-fold* OD*-characterizable for some natural number k.*

For recent results concerning the simple groups which are k-fold OD-characterizable, for $k \geqslant 2$, it was shown in [2], [25] and [26] that each of the following pairs $\{K_1, K_2\}$ of groups:

$$\begin{array}{ll}
\{A_{10}, \mathbb{Z}_3 \times J_2\}, & \\
\{B_3(5),\ C_3(5)\}, & \\
\{B_m(q),\ C_m(q)\}, & m = 2^f \geqslant 2,\ |\pi\big((q^m+1)/2\big)| = 1,\ q \text{ is an odd prime power}, \\
\{B_p(3),\ C_p(3)\}, & |\pi\big((3^p-1)/2\big)| = 1,\ p \text{ is an odd prime},
\end{array}$$

satisfy $|K_1| = |K_2|$ and $\mathrm{D}(K_1) = \mathrm{D}(K_2)$, and $h_{\rm OD}(K_i) = 2$. In general, for simple groups $B_m(q)$ and $C_m(q)$ we have

$$\big(|B_m(q)|,\ \mathrm{D}(B_m(q))\big) = \big(|C_m(q)|,\ \mathrm{D}(C_m(q))\big),$$

(see [30, Proposition 7.5]). Notice that the orthogonal group $B_n(q)$ is isomorphic to the symplectic group $C_n(q)$ when q is even, and also $B_2(q) \cong C_2(q)$ for each q. Hence, if $B_m(q)$ and $C_m(q)$ are non-isomorphic groups, then it follows that

$$h_{\rm OD}(B_m(q)) = h_{\rm OD}(C_m(q)) \geqslant 2.$$

Until recently, we do not know if there exists a non-abelian finite *simple* group which is k-fold OD-characterizable for $k \geqslant 3$. Therefore, the following problem may be of interest.

Problem 1. *Is there a non-abelian finite simple group S for which $h_{\rm OD}(S) \geqslant 3$?*

In this paper, we focus our attention on the OD-characterizability of projevtive special linear groups over binary field, that is $\mathrm{PSL}(n, 2)$, and their automorphism groups. We shortly denote $\mathrm{PSL}(n, q)$ by $L_n(q)$. Recall that $L_2(2) \cong \mathbb{S}_3$, $L_3(2) \cong L_2(7)$ and $L_4(2) \cong \mathbb{A}_8$. Clearly $s(L_2(2)) = 2$. By [16], we have $s(L_3(2)) = 3$, $s(L_4(2)) = 2$, and

$$s(L_n(2)) = \begin{cases} 1 & \text{if } n \neq p,\ p+1; \\ 2 & \text{if } n = p \text{ or } p+1, \end{cases}$$

where $p \geqslant 5$ is a prime number. More precisely, in the latter case, when $n = p$ or $p + 1$, $L_n(2)$ has two connected components, one of them is $\mathrm{GK}_1(L_n(2))$ with

$$\pi_1(L_p(2)) = \pi\Big(2\prod_{i=1}^{p-1}(2^i - 1)\Big), \quad (\text{resp. } \pi_1(L_{p+1}(2)) = \pi\Big(2(2^{p+1} - 1)\prod_{i=1}^{p-1}(2^i - 1)\Big)),$$

and the other in both cases is $\mathrm{GK}_2(L_n(2))$ with $\pi_2 = \pi(2^p - 1)$, while if $n \neq p, p+1$, then $\pi_1(L_n(2)) = \pi(L_n(2))$. The orders of finite simple groups under discussion here are:

$$|L_n(2)| = 2^{\binom{n}{2}}\prod_{i=2}^{n}(2^i - 1).$$

Previously, it was proved that many of projevtive special linear groups over binary field are OD-characterizable.

- It was proved in [3] that the linear groups $L_p(2)$ and $L_{p+1}(2)$, for which $|\pi(2^p-1)| = 1$, are OD-characterizable. Note that if $|\pi(2^p-1)| = 1$, then 2^p-1 is a prime (see [13, Ch. IX, Lemma 2.7]). A list of all known primes p such that $2^p - 1$ is also prime (which is called a Mersenne prime) is as follows: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 43112609 (see [20]). Therefore, the linear groups $L_p(2)$ and $L_{p+1}(2)$ for these primes p are OD-characterizable.

- The OD-characterizability of $L_9(2)$ was established in [14].

For the values of $|G|$, $s(G)$ and $h_{\mathrm{OD}}(G)$ for certain projective special linear groups over binary field, see Table 1.

Table 1. *The value of $h_{\mathrm{OD}}(\cdot)$ for some projective special linear groups over binary field.*

G	$\|G\|$	$s(G)$	$h_{\mathrm{OD}}(G)$	Refs.
$L_2(2) \cong \mathbb{S}_3$	$2 \cdot 3$	2	1	[23]
$L_3(2) \cong L_2(7)$	$2^3 \cdot 3 \cdot 7$	3	1	[3, 41]
$L_4(2) \cong A_8$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	1	[23]
$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2	1	[3]
$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2	1	[3]
$L_7(2)$	$2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$	2	1	[3]
$L_8(2)$	$2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127$	2	1	[3]
$L_9(2)$	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 17 \cdot 31 \cdot 73 \cdot 127$	1	1	[14]
$L_{10}(2)$	$2^{45} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127$	1	Unknown	-
$L_{11}(2)$	$2^{55} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$	2	Unknown	-

So far, we have not found any natural number $n \geqslant 2$ for which $h_{\rm OD}(L_n(2)) > 1$. On this basis, we put forward the following conjecture.

Conjecture 1.1 *The projective special linear groups $L_n(2)$ for all integers $n \geqslant 2$ are OD-characterizable.*

In this paper, we will continue to review research on this subject and we show the following result which confirms the above conjecture.

Theorem 1.2 *The projective special linear groups $L_{10}(2)$ and $L_{11}(2)$ are OD-characterizable.*

It should be mentioned that, in fact, among the finite simple groups with disconnected Gruenberg-Kegel graph, $L_{11}(2)$ is a first example of the simple OD-characterizable group S with $\Omega_0(S) = \emptyset$, whereas for all the simple OD-characterizable groups S known thus far, the set $\Omega_0(S)$ is not empty.

We now return to studying the automorphism groups of projective special linear groups over binary field. It has already been shown that the automorphism groups: $\mathrm{Aut}(L_2(2)) \cong \mathrm{Aut}(\mathbb{S}_3) \cong \mathbb{S}_3$, $\mathrm{Aut}(L_3(2)) \cong \mathrm{Aut}(L_2(7)) = \mathrm{PGL}(2,7)$, and $\mathrm{Aut}(L_4(2)) \cong \mathrm{Aut}(\mathbb{A}_8) = \mathbb{S}_8$, are OD-characterizable [3, 23, 36]. In Section 3, we will prove that the automorphism groups $\mathrm{Aut}(L_p(2))$ and $\mathrm{Aut}(L_{p+1}(2))$, where $2^p - 1 \geqslant 31$ is a Mersenne prime, are also OD-characterizable. Combining with the above results, the following theorem is derived.

Theorem 1.3 *Let $2^p - 1$ be a Mersenne prime. Then the automorphism groups $\mathrm{Aut}(L_p(2))$ and $\mathrm{Aut}(L_{p+1}(2))$ are OD-characterizable.*

Again we have not found any natural number $n \geqslant 2$ for which $h_{\rm OD}(\mathrm{Aut}(L_n(2))) \geqslant 2$. Hence, we put forward the following conjecture.

Conjecture 1.2 *The automorphism groups $\mathrm{Aut}(L_n(2))$ for all integers $n \geqslant 2$ are OD-characterizable.*

We conclude the introduction with notation to be used in the rest of this paper. Throughout, by $\mathbb{A}_n$ and $\mathbb{S}_n$, we denote the alternating and the symmetric groups on n letters, respectively. We denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G, where $p \in \pi(G)$. Moreover G_p denotes a Sylow p-subgroup of G for $p \in \pi(G)$. If H is a subgroup of G, then $N_G(H)$ is the normalizer of H in G. Given some positive integer n and some prime p, denote by n_p the p-part of n, that is the largest power of p diving n. We denote by $H : K$ (resp. $H \cdot K$) a split extension (resp. a non-split extension) of a normal subgroup H by another subgroup K. Note that, split extensions are the same as semi-direct products. All further unexplained notation is standard and refers to [7], for instance.

2 Preliminaries

Given a graph $\Gamma = (V, E)$, a set of vertices $I \subseteq V$ is said to be an independent set of Γ if no two vertices in I are adjacent in Γ. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum cardinality of an independent set among all independent sets of Γ. The following classical bound holds for every graph Γ and is due to Caro and Wei.

Lemma 2.1 ([5], [31]) *Let $\Gamma = (V, E)$ be a graph with independence number $\alpha(\Gamma)$. Then*

$$\alpha(\Gamma) \geqslant \sum_{v \in V} \frac{1}{1 + d(v)},$$

where $d(v)$ is the degree of the vertex v in Γ.

Given a group G, for convenience, we will denote $\alpha(\mathrm{GK}(G))$ as $t(G)$. Moreover, for a vertex $r \in \pi(G)$, let $t(r, G)$ denote the maximal number of vertices in independent sets of $\mathrm{GK}(G)$ containing r.

Theorem 2.1 (Theorem 1, [29]) *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) *There exists a finite non-abelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K| \cdot |\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following holds:*

(3.1) *every prime $r \in \pi(G)$ non-adjacent to 2 in $\mathrm{GK}(G)$ does not divide the product $|K| \cdot |\bar{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*

(3.2) *there exists a prime $r \in \pi(K)$ non-adjacent to 2 in $\mathrm{GK}(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong \mathbb{A}_7$ or $L_2(q)$ for some odd q.*

Lemma 2.2 (Lemma 8(1), [11]) *Let $q > 1$ be an integer, m be a natural number, and p be an odd prime. If p divides $q - 1$, then $(q^m - 1)_p = m_p \cdot (q - 1)_p$.*

Given positive integers $a \geqslant 2$ and n, we say that a prime p is a primitive prime divisor of $a^n - 1$ if $p | a^n - 1$ and $p \nmid a^k - 1$ for $1 \leqslant k < n$. We denote by $\mathrm{ppd}(a^n - 1)$ the set (depending on a and n) of all primitive prime divisors of $a^n - 1$. For example, we have $\mathrm{ppd}(13^{11} - 1) = \{23, 419, 859, 18041\}$. We recall that, by Zsigmondy's theorem [44] which is given below, the set $\mathrm{ppd}(a^n - 1)$ is non-empty if $n \neq 2, 6$.

Theorem 2.2 (Zsigmondy's Theorem) *Let a, b and n be positive integers such that $(a, b) = 1$. Then there exists a prime p with the following properties:*

- *p divides $a^n - b^n$,*
- *p does not divide $a^k - b^k$ for all $k < n$,*

with the following exceptions: $a = 2$; $b = 1$; $n = 6$ and $a + b = 2^k$; $n = 2$.

Primitive prime divisors have been applied in Finite Group Theory (see [27, 30], for example). In fact, the order of any finite simple group of Lie type S of rank n over a field $\mathrm{GF}(q)$ is equal to

$$|S| = \frac{1}{d}q^N(q^{m_1} \pm 1)(q^{m_2} \pm 1) \cdots (q^{m_n} \pm 1),$$

(see 9.4.10 and 14.3.1 in [6]). Therefore any prime divisor r of $|S|$ distinct from the characteristic p is a primitive prime divisor of $q^m - 1$, for some natural m. In particular, if $S = L_n(q)$, with $q = p^f$, then we have

$$|S| = \frac{1}{(n, q-1)}q^{\binom{n}{2}}(q^2 - 1)(q^3 - 1) \cdots (q^n - 1).$$

Now, it is easy to see that

$$\pi(S) \setminus \{p\} = \bigcup_{i=2}^{n} \mathrm{ppd}(q^i - 1).$$

The following lemma (which is an immediate corollary of [30, Propositions 2.1, 3.1 (1)]) gives the adjacency criterion for two prime divisors in the prime graph associated with a projective special linear groups $L_n(2)$.

Lemma 2.3 *Let L be the projective special linear group $L_n(2)$, with $n \geqslant 3$. Let $r, s \in \pi(L) \setminus \{2\}$ with $r \in \mathrm{ppd}(2^k - 1)$ and $s \in \mathrm{ppd}(2^l - 1)$ and assume that $2 \leqslant k \leqslant l$. Then*

(1) *r and 2 are adjacent if and only if $k \leqslant n - 2$;*

(2) *r and s are adjacent if and only if $k + l \leqslant n$ or l is divisible by k.*

In particular, every two prime divisors of $2^m - 1$, for a fixed natural number $m \leqslant n$, are adjacent in $\mathrm{GK}(L)$.

The next result which completely determines the degree of all vertices in the Gruenberg-Kegel graph $\mathrm{GK}(L_n(2))$, is a simple consequence of Lemma 2.3.

Corollary 2.1 *Let L be the projective special linear group $L_n(2)$ with $n \geqslant 3$. Let $r \in \pi(L)$ be an odd prime and $r \in \mathrm{ppd}(2^k - 1)$. Then the following hold:*

(*a*) $\deg_L(2) = |\pi(L)| - |\mathrm{ppd}(2^{n-1} - 1) \cup \mathrm{ppd}(2^n - 1)| - 1$. *In particular,* $t(2, L) \geq 2$.

(*b*) *If $k = n$ or $n - 1$, then* $\deg_L(r) = |\pi(2^k - 1)| - 1$.

(*c*) *If $k \neq n, n - 1$, then*

$$\deg_L(r) = \begin{cases} |\bigcup\limits_{i=2}^{n-k} \mathrm{ppd}(2^i - 1)| + |\mathrm{ppd}(2^{[\frac{n}{k}]k} - 1)| & k \leqslant n/2, \\ |\pi(2^k - 1) \cup \bigcup\limits_{i=2}^{n-k} \mathrm{ppd}(2^i - 1)| & k > n/2. \end{cases}$$

Proof. Recall that, the order of L is equal to

$$|L| = 2^{\binom{n}{2}}(2^2-1)(2^3-1)\cdots(2^{n-1}-1)(2^n-1).$$

Therefore any odd prime divisor r of $|L|$ is a primitive divisor of 2^m-1, for some natural number $m \leqslant n$.

(a) By Lemma 2.3 (1), we have $2 \sim r$ if and only if $k \leqslant n-2$. Therefore, we obtain

$$\deg_L(2) = |\pi(L)| - |\mathrm{ppd}(2^{n-1}-1) \cup \mathrm{ppd}(2^n-1)| - 1.$$

In addition, since

$$(2^{n-1}-1, 2^n-1) = 2^{(n-1,n)} - 1 = 1,$$

and by Theorem 2.2, we get

$$|\mathrm{ppd}(2^{n-1}-1) \cup \mathrm{ppd}(2^n-1)| \geqslant 2.$$

Therefore, we obtain $\deg_L(2) \leqslant |\pi(L)| - 3$, which forces $t(2, L) \geqslant 2$, as required.

(b) If $k = n$ or $n-1$, then by Lemma 2.3 (1), $2 \nsim r$, and if $s \in \pi(L) \setminus \{2, r\}$ with $s \in \mathrm{ppd}(2^l-1)$, then by Lemma 2.3 (2), $s \sim r$ if and only if l divides k. But then 2^l-1 divides 2^k-1, and so $s \in \pi(2^k-1)$. Finally, in both cases, we obtain $\deg_L(r) = |\pi(2^k-1)|-1$.

(c) The conclusion follows immediately from Lemma 2.3. $\square$

We are now able to compute the degree pattern of simple group $L_n(2)$, for a fixed n.

Table 2. *The degree pattern of some linear groups $L_n(2)$.*

$L_n(2)$	$D(L_n(2))$
$L_2(2)$	$(0,0)$
$L_3(2)$	$(0,0,0)$
$L_4(2)$	$(1,2,1,0)$
$L_5(2)$	$(2,3,1,2,0)$
$L_6(2)$	$(3,3,2,2,0)$
$L_7(2)$	$(4,4,3,3,2,0)$
$L_8(2)$	$(4,5,4,4,2,3,0)$
$L_9(2)$	$(5,6,5,5,2,4,1,2)$
$L_{10}(2)$	$(6,7,5,6,2,3,5,1,3)$
$L_{11}(2)$	$(7,8,6,7,2,4,1,5,3,1,4)$
$L_{12}(2)$	$(8,9,7,8,3,3,4,1,6,3,1,5)$
$L_{13}(2)$	$(10,11,8,9,4,3,5,3,7,4,3,5,0)$
$L_{14}(2)$	$(11,12,9,11,5,4,5,4,8,2,5,4,6,0)$
$L_{15}(2)$	$(12,13,11,12,5,4,6,5,9,2,5,5,7,2,2)$
$L_{16}(2)$	$(13,14,12,13,5,4,7,6,11,3,6,6,8,2,3,3)$
$L_{17}(2)$	$(14,15,13,14,6,5,8,6,12,4,7,6,9,4,3,4,0)$
$L_{18}(2)$	$(15,16,14,15,7,5,9,3,7,13,5,8,7,11,4,4,5,0)$
$L_{19}(2)$	$(16,17,15,16,8,6,11,3,8,14,6,9,8,12,5,5,5,2,0)$
$L_{20}(2)$	$(17,18,16,17,9,7,12,4,9,15,4,6,11,9,13,5,5,6,3,0)$

It may be finally worth noting that $L_n(2) \hookrightarrow L_{n+1}(2)$, which implies that:

- If $n \neq 5$, then $\pi(L_n(2)) \subsetneqq \pi(L_{n+1}(2))$ and $\pi_e(L_n(2)) \subsetneqq \pi_e(L_{n+1}(2))$. Moreover, we have $\pi(L_5(2)) = \pi(L_6(2))$, while $\pi_e(L_5(2)) \subsetneqq \pi_e(L_6(2))$.
- The Gruenberg-Kegel graph $\mathrm{GK}(L_n(2))$ is a subgraph of $\mathrm{GK}(L_{n+1}(2))$,
- If $p \in \pi(L_n(2))$, then $\deg_{L_n(2)}(p) \leqslant \deg_{L_{n+1}(2)}(p)$.

The following lemma (which is taken from [18, Lemma 8]) shows that none of the sets of "generalized nonnegative matrices" which we mentioned in Sections 1 and 2 is a convex set.

Lemma 2.4 (Lemma 8, [18]) *Let G be a group. If $t(G) \geqslant 3$, then G is non-solvable.*

Lemma 2.5 *Let p be an odd prime and $L \in \{L_p(2), L_{p+1}(2)\}$. Suppose G is a finite group which satisfies the conditions $|G| = |L|$ and $D(G) = D(L)$. Then there hold.*

(a) *There exist three primes in $\pi(G)$ pairwise non-adjacent in $\mathrm{GK}(G)$, that is $t(G) \geqslant 3$. In particular, G is a non-solvable group.*

(b) *There exists an odd prime in $\pi(G)$ which is not adjacent to the prime 2 in $\mathrm{GK}(G)$; that is $t(2, G) \geqslant 2$.*

(c) *There exists a finite non-abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G. Furthermore, $t(S) \geqslant t(G) - 1$.*

Proof. (a) Suppose first that $L = L_p(2)$. If $p = 3$ (resp. 5, 7), then the set $\{2, 3, 5\}$ (resp. $\{5, 7, 31\}$, $\{7, 31, 127\}$) is an independent set in $\mathrm{GK}(G)$, and hence $t(G) \geqslant 3$. Therefore, we may assume that $p \geqslant 11$.

Assume to the contrary that $t(G) \leqslant 2$. We now point out some elementary facts about the degree of vertices in $\mathrm{GK}(G)$. Firstly, with a similar argument, as in the proof of Proposition 2.1 in [30], we can verify that

$$\mathrm{ppd}(2^p - 1) = \pi(2^p - 1).$$

Secondly, for two non-adjacent vertices $p_1, p_2 \in \pi(G)$, since $t(G) \leqslant 2$, we obtain

$$\deg_G(p_1) + \deg_G(p_2) \geqslant |\pi(G)| - 2. \tag{1}$$

In what follows, for the sake of convenience, we put $|\mathrm{ppd}(2^p - 1)| = m$ and $|\pi(G)| = n$. We now consider two cases separately.

Case 1. $m = 1$. Suppose that $\pi(2^p - 1) = \{q\}$. Then $\deg_G(q) = 0$, and so the Gruenberg-Kegel graph $\mathrm{GK}(G)$ is not connected. On the other hand, by Corollary 2.1 (a) and Theorem 2.2, we obtain

$$\deg_G(2) = n - |\mathrm{ppd}(2^{p-1} - 1)| - 2 \leqslant n - 3.$$

Hence, there exists a prime $q' \in \pi(G) \setminus \{q\}$ such that $q' \nsim 2$. Therefore, the set $\{2, q', q\}$ is an independent set in $\mathrm{GK}(G)$, against our hypothesis.

Case 2. $m \geqslant 2$. Suppose that $\mathrm{ppd}(2^p-1)=\{p_1,p_2,\ldots,p_m\}$. If there exists p_i such that $2 \nsim p_i$, then, from Eq. (1), we conclude that

$$\deg_G(2)+\deg_G(p_i) \geqslant n-2.$$

Applying Corollary 2.1 $(a),(b)$ and some simplification this leads to

$$n-|\pi(2^{p-1}-1)| \geqslant n,$$

which is a contradiction. Therefore, we may assume that $2 \sim p_i$, for each $i=1,2,\ldots,m$, and so $\deg_G(2) \geqslant m$. Now we apply Corollary 2.1 (a), to get

$$n-m-1 > n-m-|\mathrm{ppd}(2^{p-1}-1)|-1=\deg_G(2) \geqslant m,$$

or equivalently, $m < \frac{n-1}{2}$. Furthermore, there are two primes p_i, p_j such that $p_i \nsim p_j$ in $\mathrm{GK}(G)$, otherwise $\deg_G(p_i) \geqslant m$ and this contradicts the fact that $\deg_G(p_i)=m-1$. Again, by Eq. (1),

$$\deg_G(p_i)+\deg_G(p_j) \geqslant n-2,$$

which forces $m \geqslant \frac{n}{2}$, a contradiction. This completes the proof for $L=L_p(2)$.

In the case when $L=L_{p+1}(2)$, the proof is similar to the previous case and, therefore, omitted. Finally, in both cases, $t(G) \geqslant 3$ and by Lemma 2.4, G is a non-solvable group.

(b) It is obvious, because $\deg_G(2) \leqslant |\pi(G)|-3$.

(c) Follows from (a), (b) and Theorem 2.1. □

Proposition 2.1 (Theorem A, [32]) *If G is a finite group with disconnected Gruenberg-Kegel graph* $\mathrm{GK}(G)$*, then one of the following holds:*

(a) *$s(G)=2$, G is a Frobenius group.*

(b) *$s(G)=2$, $G=ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are Frobenius groups (such a group G is called a 2-Frobenius group).*

(c) *There exists a non-abelian simple group P such that $P \leqslant G/K \leqslant \mathrm{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup K of G, and G/P is a $\pi_1(G)$-group. Moreover, $\mathrm{GK}(P)$ is disconnected, $s(P) \geqslant s(G)$.*

The following Propositions deal with the structure of Frobenius and 2-Frobenius groups and their Gruenberg-Kegel graphs. One may find their proofs in [9, 19].

Proposition 2.2 (Theorem 3.1, [9]) *If G is a Frobenius group with the kernel K and complement C, then the following conditions hold:*

(1) *K is nilpotent and so its Gruenberg-Kegel graph $\mathrm{GK}(K)$ is a complete graph, that is $\mathrm{GK}(K)=K_{|\pi(K)|}$;*

(2) *$s(G) = 2$ and the connected components of $\mathrm{GK}(G)$ are $\mathrm{GK}(K)$ and $\mathrm{GK}(C)$, that is, $\mathrm{GK}(G) = \mathrm{GK}(K) \oplus \mathrm{GK}(C)$. In particular, we have $\mathrm{OC}(G) = \{|K|, |C|\}$.*

(3) *$|C|$ divides $|K| - 1$, and so $|C| < |K|$.*

Proposition 2.3 (Lemma 7, [19]) *In case* (b) *of Proposition* 2.1*:*

(1) *C and B are cyclic groups, and $|B|$ is odd;*

(2) *$\mathrm{GK}(B)$ and $\mathrm{GK}(AC)$ are connected components of the prime graph $\mathrm{GK}(G)$, and both of them are complete graphs. Hence, we have*

$$\mathrm{GK}(G) = \mathrm{GK}(AC) \oplus \mathrm{GK}(B) = K_{|\pi(AC)|} \oplus K_{|\pi(B)|}.$$

In particular, $s(G) = 2$, $\pi_1(G) = \pi(AC), \pi_2(G) = \pi(B)$, $\mathrm{OC}(G) = \{|AC|, |B|\}$, and for every primes $p \in \pi(G)$, we have $\deg_G(p) = |\pi(AC)| - 1$ or $|\pi(B)| - 1$.

The following result will be used frequently throughout next section.

Lemma 2.6 *Let G be a finite group and K be a normal solvable subgroup of G. Let $p, q \in \pi(G)$ such that $p \not\equiv 1 \pmod q$, $q \not\equiv 1 \pmod p$ and $|G_pG_q| = pq$. If p divides the order of K, then $p \sim q$ in $\mathrm{GK}(G)$.*

Proof. If $q \in \pi(K)$, then K contains a cyclic subgroup of order pq, and the result is proved. Hence, we may assume that $q \notin \pi(K)$. Let P be a Sylow p-subgroup of K. Then $G = KN_G(P)$ by Frattini argument, and so $N_G(P)$ contains an element of order q, say x. Clearly $P\langle x\rangle$ is a cyclic subgroup of G of order pq, and hence $p \sim q$ in $\mathrm{GK}(G)$. This completes the proof. $\square$

We now present the following useful degree criterion for non-solvability a group using whose degree pattern.

Lemma 2.7 *Let G be a finite group satisfies $\Omega_0(G) \neq \emptyset$ and $\Omega_i(G) \neq \emptyset$ for some $1 \leqslant i \leqslant |\pi(G)| - 3$ (i.e., there exists a vertex in $\mathrm{GK}(G)$ of degree at most $|\pi(G)| - 3$). Then $t(G) \geqslant 3$ and especially G is non-solvable.*

We omit the straightforward proof.

Lemma 2.8 *([22], [35]) Let S be a finite non-abelian simple group such that its order divides $|L_n(2)|$ where $n \in \{10, 11\}$. Then*

(1) *if $n = 10$ and $\{11, 73\} \subset \pi(S)$, then S is isomorphic to $L_{10}(2)$*

(2) *If $n = 11$ and $\{23, 89\} \subset \pi(S)$, then S is isomorphic to $L_{11}(2)$.*

Proof. By results collected in [22, 35], if S is a finite non-abelian simple group such that its order divides the order of $L_{11}(2)$, then S is isomorphic to one of the simple groups listed below in Table 3. Now, the lemma follows by checking the conditions in (1) and (2). $\square$

Table 3. *The simple group S whose order divides* $|L_{11}(2)| = 2^{55} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$.

S	$\|S\|$	S	$\|S\|$
$\mathbb{A}_5$	$2^2 \cdot 3 \cdot 5$	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
$\mathbb{A}_6$	$2^3 \cdot 3^2 \cdot 5$	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	$L_4(2^2)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$
$\mathbb{A}_7$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$C_2(2^2)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$
$\mathbb{A}_8$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$L_2(2^4)$	$2^4 \cdot 3 \cdot 5 \cdot 17$
$\mathbb{A}_9$	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$
$\mathbb{A}_{10}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
$B_3(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
$L_3(2^2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
$L_2(2^3)$	$2^3 \cdot 3^2 \cdot 7$	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	$S_{10}(2)$	$2^{25} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31$
$L_2(7^2)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	$O_{10}^+(2)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$L_5(2^2)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31$
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	$L_2(2^5)$	$2^5 \cdot 3 \cdot 11 \cdot 31$
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	$L_3(2^3)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	$U_3(3^2)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$L_2(89)$	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 89$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$L_7(2)$	$2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$L_8(2)$	$2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127$
$\mathbb{A}_{11}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$L_9(2)$	$2^{36} \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 17 \cdot 31 \cdot 73 \cdot 127$
$\mathbb{A}_{12}$	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	$L_{10}(2)$	$2^{45} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127$
$C_4(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	$L_{11}(2)$	$2^{55} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$

3 OD-Characterizability of Certain Groups

As we mentioned earlier in the Introduction, we are going to show that the simple groups $L_{10}(2)$, $L_{11}(2)$ and the automorphism groups $\mathrm{Aut}(L_p(2))$ and $\mathrm{Aut}(L_{p+1}(2))$, where $2^p - 1$ is a Mersenne prime, are uniquely determined through their orders and degree patterns.

3.1 OD-Characterizability of Simple Groups $L_{10}(2)$ and $L_{11}(2)$

Here, we show that the simple groups $L_{10}(2)$ and $L_{11}(2)$ are OD-characterizable. We start with the following theorem.

Theorem 3.1 *Let G be a finite group which satisfies the following conditions:*

- $|G| = 2^{45} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127$, *and*
- $D(G) = (6, 7, 5, 6, 2, 3, 5, 1, 3)$.

Then G is isomorphic to $L_{10}(2)$.

Proof. Applying Lemma 2.1 and easy computations show that

$$t(G) \geqslant \sum_{p\in\pi(G)} \frac{1}{1+\deg_G(p)} \approx 2.07.$$

Hence, $t(G) \geqslant 3$ and G is a non-solvable group by Lemma 2.4. In addition, since $\deg_G(2) = |\pi(G)| - 3 = 6$, $t(2, G) \geqslant 2$. Let K be the maximal normal solvable subgroup of G. Then, by Theorem 2.1, there exists a finite non-abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$. Evidently, K is a $\{11, 73\}'$-group, since otherwise by Lemma 2.6, we obtain $\deg_G(11) \geqslant 3$ or $\deg_G(73) \geqslant 3$, which is a contradiction. Now, it is clear that $|S|$ is divisible by 11 and 73, and from Lemma 2.8 (1), it follows that $S \cong L_{10}(2)$. Finally, since $|G| = |L_{10}(2)|$, we conclude that $|K| = 1$ and $G \cong L_{10}(2)$. □

Theorem 3.2 *Let G be a finite group. Then $G \cong L_{11}(2)$ if and only if G satisfies the following conditions:*

- $|G| = 2^{55} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 31^2 \cdot 73 \cdot 89 \cdot 127$, *and*
- $D(G) = (7, 8, 6, 7, 2, 4, 1, 5, 3, 1, 4)$.

Proof. First of all, it follows from Lemma 2.5 (a), $t(G) \geqslant 3$ and G is a non-solvable group. Moreover, since $\deg_G(2) = |\pi(G)| - 4 = 7$, $t(2, G) \geqslant 2$. Let K be the maximal normal solvable subgroup of G. Then, by Theorem 2.1, there exists a finite non-abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$. Moreover, one can easily see that K is a $\{23, 89\}'$-group, which follows directly from Lemma 2.6 and the facts that $\deg(23) = \deg(89) = 1$. Now, it is clear that $|S|$ is divisible by 23 and 89. Using Lemma 2.8 (2), it follows that $S \cong L_{11}(2)$. Finally, since $L_{11}(2) \leqslant G/K \leqslant \mathrm{Aut}(L_{11}(2))$ and $|G| = |L_{11}(2)|$, we deduce that $|K| = 1$ and $G \cong L_{11}(2)$. □

3.2 On the Automorphism Group of $L_n(2)$

It is known (see [10, Theorem 2.5.12]) that, the group of outer automorphisms of a simple group of Lie type is generated by the diagonal automorphisms, the graph automorphisms (of the underlying Dynkin diagram), and the field automorphisms of the field of definition. Especially, for $S = L_n(q)$, with $n \geqslant 2$ and $q = p^f$, we have (see also [7]):

$$|\mathrm{Out}(S)| = (n, q-1) \cdot f \cdot 2.$$

Therefore, the only outer automorphism of simple group $L_n(2)$, $n \geq 3$, is the graph automorphism of order 2, corresponds to the symmetry of its Dynkin diagram. We denote by σ this automorphism and set $L := L_n(2)$. Then, we have $\mathrm{Aut}(L) = L \cdot \langle \sigma \rangle$, and so $|\mathrm{Aut}(L) : L| = 2$. The following general results may be stated:

Lemma 3.1 *Let S be a simple group with $|\mathrm{Aut}(S) : S| = 2$. Then there holds:*

$$\mathrm{GK}(\mathrm{Aut}(S)) - \{2\} = \mathrm{GK}(S) - \{2\}.$$

In particular, if $r \in \pi(S) - \{2\}$, then $\deg_S(r) \leqslant \deg_{\mathrm{Aut}(S)}(r) \leqslant \deg_S(r) + 1$, and in addition, if $2 \sim r$ in $\mathrm{GK}(S)$, then $\deg_{\mathrm{Aut}(S)}(r) = \deg_S(r)$.

Proof. First of all, we note that $S \cong \mathrm{Inn}(S) \leqslant \mathrm{Aut}(S)$, and so $\pi_e(S) \subseteq \pi_e(\mathrm{Aut}(S))$ and $\mathrm{GK}(S)$ is a subgraph of $\mathrm{GK}(\mathrm{Aut}(S))$. We claim that $\pi_e(\mathrm{Aut}(S)) \setminus \pi_e(S)$ is a subset of the set of even natural numbers. Suppose $m \in \pi_e(\mathrm{Aut}(S)) \setminus \pi_e(S)$ is an odd number. Then there exists $x \in \mathrm{Aut}(S) \setminus S$ such that $o(x) = m$. On the other hand, we have $x^{-1} = x^{m-1} \in S$, since $|\mathrm{Aut}(S) : S| = 2$ and $m-1$ is even. Hence $x \in S$, which is a contradiction.

Notice that $\pi(\mathrm{Aut}(S)) = \pi(S)$. In what follows we claim that if p and q are two odd primes such that $p \nsim q$ in $\mathrm{GK}(S)$, then $p \nsim q$ in $\mathrm{GK}(\mathrm{Aut}(S))$. Assume that the claim is false and $p \sim q$ in $\mathrm{GK}(\mathrm{Aut}(S))$. Then S dose not contain an element of order pq, while from the previous paragraph of the proof the automorphism group $\mathrm{Aut}(S)$ has an element of order $2pq$, say x. Therefore $x^2 \in S$ and $o(x^2) = pq$, which is a contradiction. $\square$

A sequence of non-negative integers $(a_1, a_2, \ldots, a_k)$ is said to be *majorised* by another such sequence $(b_1, b_2, \ldots, b_k)$ if $a_i \leqslant b_i$ for $1 \leqslant i \leqslant k$. A graph Γ_1 is *degree-majorised* by a graph Γ_2 if $V(\Gamma_1) = V(\Gamma_2)$ and the *non-ascending degree sequence* of Γ_1 is majorised by that of Γ_2. By Lemma 3.1, we have immediately the following:

Corollary 3.1 *Let S be a simple group with $|\mathrm{Aut}(S) : S| = 2$. Then $\mathrm{GK}(S)$ is degree-majorised by $\mathrm{GK}(\mathrm{Aut}(S))$.*

Hereinafter, we assume that $L := L_n(2)$ with $n \in \{p, p+1\}$, where p is an odd prime. We list some elementary properties of the automorphism group $\mathrm{Aut}(L)$ that are useful in the following:

- $|\mathrm{Aut}(L)| = 2 \cdot |L| = 2^{\binom{n}{2}+1}(2^2-1)(2^3-1)\cdots(2^n-1)$ and $\pi(\mathrm{Aut}(L)) = \pi(L)$.
- $s(\mathrm{Aut}(L)) = 2$ (see [8, Lemma 2.2]).

- $\pi_1(\mathrm{Aut}(L)) = \pi_1(L)$ and $\pi_2(\mathrm{Aut}(L)) = \pi_2(L) = \pi(2^p - 1)$. In fact, if $n = p$, then

  $$C_L(\sigma) \cong PSO^+(p, 2) \ \text{ of order } \ 2^{((p-1)/2)^2}(2^2 - 1)(2^4 - 1)\cdots(2^{p-1} - 1),$$

  and if $n = p + 1$, then

  $$C_L(\sigma) \cong PSp(p + 1, 2) \ \text{ of order } \ 2^{((p+1)/2)^2}(2^2 - 1)(2^4 - 1)\cdots(2^{p-1} - 1)(2^{p+1} - 1),$$

  (see [4, 19.9]). Therefore, if $q \in \mathrm{ppd}(2^p - 1)$, then $q \nsim 2$ in $\mathrm{GK}(\mathrm{Aut}(L))$. Moreover, by Lemma 3.1 and the fact that $\pi_2(L) = \pi(2^p - 1)$, q is not adjacent to any odd primes in $\pi_1(L) \setminus \pi(2^p - 1)$.

In the sequel, we will show that the automorphism group of linear groups $L_p(2)$ and $L_{p+1}(2)$, where $2^p - 1$ is a Mersenne prime, are uniquely determined through their orders and degree patterns. We start with the following lemmas.

Lemma 3.2 *Let $n \geqslant 3$ be an integer and $L = L_n(2)$. Then there hold.*

(1) *If $n \geqslant 12$ is even, then $(2^k - 1)^2$, $2 \leqslant k \leqslant n$, does not divide the order of* $\mathrm{Aut}(L)$ *if and only if $k = \frac{n}{2} + i$, $i = 1, 2, \ldots, \frac{n}{2}$.*

(2) *If $n \geqslant 13$ is odd, then $(2^k - 1)^2$, $2 \leqslant k \leqslant n$, does not divide the order of* $\mathrm{Aut}(L)$ *if and only if $k = \frac{n-1}{2} + i$, $i = 1, 2, \ldots, \frac{n+1}{2}$.*

(3) *If $n \leqslant 11$, then $(2^k - 1)^2$ does not divide the order of* $\mathrm{Aut}(L)$ *if and only if one of the following statements holds:*

  (3.1) $n = 11$ *and* $k = 7, 8, 9, 10, 11$.

  (3.2) $n = 10$ *and* $k = 7, 8, 9, 10$.

  (3.3) $n = 9$ *and* $k = 5, 7, 8, 9$.

  (3.4) $n = 8$ *and* $k = 5, 7, 8$.

  (3.5) $n = 7$ *and* $k = 4, 5, 7$.

  (3.6) $n = 6$ *and* $k = 4, 5$.

  (3.7) $n = 5$ *and* $k = 3, 4, 5$.

  (3.8) $n = 4$ *and* $k = 3, 4$.

  (3.9) $n = 3$ *and* $k = 2, 3$.

Proof. Since the proofs of (1) and (2) are similar, only the proof for (1) is presented. The proof of (3) is a straightforward verification. First of all, we recall that

$$|\mathrm{Aut}(L)| = 2 \cdot |L| = 2^{\binom{n}{2}+1} \prod_{i=2}^{n} (2^i - 1),$$

because $|\mathrm{Out}(L)| = 2$. Moreover, if $s \in \mathrm{ppd}(2^k - 1)$, then $s | 2^l - 1$ if and only if k divides l (see the proof of Proposition 2.1 in [30]). Assume first that $k \geqslant \frac{n}{2} + 1 \geqslant 7$. Applying

Theorem 2.2, we can consider a primitive prime divisor $s \in \mathrm{ppd}(2^k-1)$, and suppose that $s^m \parallel 2^k-1$. As we mentioned before, if $s|2^l-1$, then k divides l, and hence $l \geqslant 2k \geqslant n+2$, which means that $(s, |\mathrm{Aut}(L)|/(2^k-1)) = 1$, and so $s^m \parallel |\mathrm{Aut}(L)|$. Hence, if $(2^k-1)^2$ divides $|\mathrm{Aut}(L)|$ then we must have $s^{2m} \big| |\mathrm{Aut}(L)|$, which is a contradiction. Assume next that $k \leqslant \frac{n}{2}$. In this case $2k \leqslant n$, and since $2^k-1|2^{2k}-1$, it follows that $(2^k-1)^2 \big| |\mathrm{Aut}(L)|$. This completes the proof of (1). $\square$

Lemma 3.3 *Let $2^p-1 \geqslant 31$ be a Mersenne prime and $L \in \{L_p(2), L_{p+1}(2)\}$. Suppose that G is a finite group satisfies the conditions: $|G| = |\mathrm{Aut}(L)|$ and $D(G) = D(\mathrm{Aut}(L))$. Then $t(G) \geqslant 3$. In particular, G is a non-solvable group.*

Proof. We recall that $\deg_G(2^p-1) = 0$, and so $\Omega_0(G) \neq \emptyset$. To complete the proof, from Lemma 2.7, it is enough to show that $\Omega_i(G) \neq \emptyset$ for some $1 \leqslant i \leqslant |\pi(G)|-3$. If $p=5$ (resp. 7), then $\deg_{L_5(2)}(5) = 1$ and $\deg_{L_6(2)}(5) = 2$ (resp. $\deg_{L_7(2)}(31) = 2$ and $\deg_{L_8(2)}(17) = 2$). Hence, by Lemma 3.1, we have

$$\begin{array}{ll} L = L_5(2) & \deg_G(5) = \deg_{\mathrm{Aut}(L)}(5) \leqslant \deg_L(5)+1 = 2 = |\pi(G)|-3, \\ L = L_6(2) & \deg_G(5) = \deg_{\mathrm{Aut}(L)}(5) = \deg_L(5) = 2 = |\pi(G)|-3, \\ & \quad \text{(note that } 2 \sim 5 \text{ in } \mathrm{GK}(L)), \\ L = L_7(2) & \deg_G(31) = \deg_{\mathrm{Aut}(L)}(31) \leqslant \deg_L(31)+1 = 3 = |\pi(G)|-3, \\ L = L_8(2) & \deg_G(17) = \deg_{\mathrm{Aut}(L)}(17) \leqslant \deg_L(17)+1 = 3 = |\pi(G)|-3, \end{array}$$

as required.

Therefore, we may assume that $p \geqslant 13$. In this case, we consider a primitive prime divisor of $2^{p-1}-1$, say r. By Lemma 2.3, one can easily see that $r \nsim s$ in $\mathrm{GK}(L)$, for each

$$s \in \bigcup_{i=1}^{\frac{p-3}{2}} \mathrm{ppd}(2^{\frac{p-1}{2}+i}-1).$$

Hence, by Theorem 2.2, we obtain

$$\deg_L(r) \leqslant |\pi(L)| - |\bigcup_{i=1}^{\frac{p-3}{2}} \mathrm{ppd}(2^{\frac{p-1}{2}+i}-1)| - 1 \leqslant |\pi(L)| - \frac{p-3}{2} - 1 \leqslant |\pi(L)| - 6,$$

because $p \geqslant 13$. Finally, we conclude that

$$\deg_G(r) = \deg_{\mathrm{Aut}(L)}(r) \leqslant \deg_L(r)+1 \leqslant |\pi(L)|-5 = |\pi(G)|-5,$$

as required. $\square$

We are now ready to prove our main result.

Theorem 3.3 *Let 2^p-1 be a Mersenne prime and $L \in \{L_p(2), L_{p+1}(2)\}$. Suppose that G is a finite group satisfies the conditions: $|G| = |\mathrm{Aut}(L)|$ and $D(G) = D(\mathrm{Aut}(L))$. Then G is isomorphic to* $\mathrm{Aut}(L)$.

Proof. First of all, we consider the cases $p = 2$ and 3. Indeed, if $L = L_2(2) \cong \mathbb{S}_3$, then $\mathrm{Aut}(L) \cong L$ and the result now follows by applying Theorem 1.2 in [3]. If $L = L_3(2)$, then $\mathrm{Aut}(L) = \mathrm{PGL}(2, 7)$ and the result is proved in [36]. Finally, if $L = L_4(2) \cong \mathbb{A}_8$, then $\mathrm{Aut}(L) \cong \mathbb{S}_8$ and the result follows from Theorem 1.5 in [23].

Therefore, we assume that 2^p-1 is a Mersenne prime with $p \geqslant 5$ and $L \in \{L_p(2), L_{p+1}(2)\}$. Let G be a finite group with $|G| = |\mathrm{Aut}(L)|$ and $D(G) = D(\mathrm{Aut}(L))$. Then $\pi(G) = \pi(\mathrm{Aut}(L)) = \pi(L)$, $2^p - 1$ is the largest prime in $\pi(G)$ and $\deg_G(2^p - 1) = 0$. Moreover, by Corollary 2.1 (c), $\deg_G(3) = |\pi(G)| - 2$, which forces $s(G) = 2$. More precisely, we have

$$\pi_1(G) = \pi_1(L) \quad \text{and} \quad \pi_2(G) = \{2^p - 1\},$$

and since G and $\mathrm{Aut}(L)$ have the same order, we conclude that

$$\mathrm{OC}(G) = \mathrm{OC}(\mathrm{Aut}(L)) = \{m_1, m_2\},$$

where

$$m_1 = \begin{cases} 2^{\binom{p}{2}+1}(2^2-1)(2^3-1)\cdots(2^{p-1}-1) & \text{if} \quad L = L_p(2), \\ 2^{\binom{p+1}{2}+1}(2^2-1)(2^3-1)\cdots(2^{p-1}-1)(2^{p+1}-1) & \text{if} \quad L = L_{p+1}(2); \end{cases}$$

and

$$m_2 = 2^p - 1.$$

Furthermore, by Proposition 2.1, one of the following cases holds:

Case 1. G is either a Frobenius group or a 2-Frobenius group;

Case 2. There exists a non-abelian simple group P such that $P \leqslant G/K \leqslant \mathrm{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup K of G, and G/P is a $\pi_1(G)$-group. Moreover, $s(P) \geqslant 2$ and $\pi_2(G) = \{2^p - 1\}$.

In what follows, we will consider every case separately.

Lemma 3.4 *Case* 1 *is impossible.*

Proof. First of all, by Lemma 3.3, G is a non-solvable group. Hence, G is not a 2-Frobenius group. Assume now that G is a Frobenius group with kernel K and complement C. Then by Proposition 2.2, $\mathrm{OC}(G) = \{|K|, |C|\}$. From $|C| < |K|$ we can easily conclude that $|K| = m_1$ and $|C| = m_2 = 2^p - 1$. But then, the degree pattern of G has the following form:

$$D(G) = (n-2, n-2, \ldots, n-2, 0),$$

where $n = |\pi(G)|$, and hence $t(G) = 2$, which contradicts Lemma 3.3. $\square$

Thus Case 2 holds, that is, there exists a non-abelian simple group P such that

$$P \leqslant G/K \leqslant \mathrm{Aut}(P),$$

for some nilpotent normal $\pi_1(G)$-subgroup K of G, and G/P is a $\pi_1(G)$-group. Evidently $\pi_2(P) = \{2^p - 1\}$ and $\pi_e(P) \subseteq \pi_e(G/K) \subseteq \pi_e(G)$. Therefore, for every prime $r \in \pi(P)$, we have $\deg_P(r) \leqslant \deg_G(r)$.

Lemma 3.5 *P is isomorphic to L.*

Proof. According to the classification of the finite simple groups we know that the possibilities for P are: alternating groups $\mathbb{A}_m$, $m \geqslant 5$; 26 sporadic finite simple groups; simple groups of Lie type. We deal with the above cases separately. We will use the results summarized in Tables 1, 2 and 3 in [17].

First, suppose P is an alternating group $\mathbb{A}_m$, $m \geqslant 5$. Since $2^p - 1 \in \pi(P)$, $m \geqslant 2^p - 1$. Now, we consider a prime r between $2^{p-1} - 1$ and $2^p - 1$. It is clear that $r \in \pi(\mathbb{A}_m) \backslash \pi(G)$, but this is impossible.

Next, suppose P is a sporadic simple group. Since the odd order components of a sporadic simple group are prime less than 71, it follows that $2^p - 1 < 71$. Hence we obtain that $p = 3$ or 5. Using the results summarized in Tables 1, 2 and 3 in [17], we see that P cannot isomorphic to any sporadic simple group.

Finally, suppose P is a simple group of Lie type. Here, according to the number of the prime graph components of P, we proceed case by case analysis.

Case 3.1 $s(P) = 2$.

In this case we have $m_2(P) = 2^p - 1$.

(1) *The simple group P is isomorphic to none of the simple groups $C_n(q)$, $n = 2^m \geqslant 2$; $D_r(q)$, $r \geqslant 5$, $q = 2, 3, 5$; $D_{r+1}(q)$, $q = 2, 3$; $F_4(q)$, q odd; $G_2(q)$, $q \equiv \pm 1 \pmod 3$; ${}^2D_r(3)$, $r \geqslant 5$, $r \neq 2^n + 1$; ${}^2D_n(2)$, $n = 2^m + 1 \geqslant 5$; ${}^2D_n(3)$, $9 \leqslant n = 2^m + 1 \neq r$; ${}^3D_4(q)$, $C_r(3)$, $B_r(3)$; $B_n(q)$, $n = 2^m \geqslant 4$, q odd, ${}^2A_3(2)$ and ${}^2F_4(2)'$.*

(1.1) If $P \cong C_n(q)$, $n = 2^m \geqslant 2$, then

$$|P| = |C_n(q)| = m_1 \times m_2 = q^{n^2}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1) \times \frac{q^n + 1}{2}.$$

Because $\frac{q^n+1}{2} = 2^p - 1$, it implies that $q^n - 1 = 4(2^{p-1} - 1)$. Evidently $p \neq 7$. On the other hand, since $(q^n - 1)^2$ divides $|P|$, thus $(2^{p-1} - 1)^2$ must divides $|G|$. This is a contradiction by Lemma 3.2.

(1.2) If $P \cong D_r(q)$, $r \geqslant 5$, $q = 2, 3, 5$, then

$$|P| = |D_r(q)| = m_1 \times m_2 = q^{r(r-1)}\prod_{i=1}^{r-1}(q^{2i} - 1) \times \frac{q^r - 1}{(q-1, 4)} .$$

In this case we have $\frac{q^r-1}{(q-1,4)} = 2^p - 1$. If $q = 2$, then $2^r - 1 = 2^p - 1$, and hence $r = p$. Thus $2^{p(p-1)}$ divides $|P|$ and so $|G|$, which is a contradiction. If $q = 3$, then we obtain $2^2(2^{p-1} - 1) = 3(3^{r-1} - 1)$ and if $q = 5$ then we get $2^3(2^{p-1} - 1) = 5(5^{r-1} - 1)$. In both cases, we easy to see that $(2^{p-1} - 1)^2 \big| |G|$, which contradicts Lemma 3.2.

(1.3) If $P \cong F_4(q)$, q odd, then we have

$$|P| = |F_4(q)| = m_1 \times m_2 = q^{24}(q^4 - 1)(q^6 - 1)^2(q^8 - 1) \times (q^4 - q^2 + 1).$$

Now, from $q^4 - q^2 + 1 = 2^p - 1$ we deduce that $2(2^{p-1} - 1) = q^2(q^2 - 1)$. But then, we have $(2^{p-1} - 1)^2 \big| |G|$, which is again a contradiction by Lemma 3.2.

The other cases are settled similarly.

(2) *The simple group P is isomorphic to none of the simple groups ${}^2D_n(q)$, $n = 2^m \geqslant 4$, and $C_r(2)$.*

(2.1) If $P \cong {}^2D_n(q), n = 2^m \geqslant 4$, then

$$|P| = |{}^2D_n(q)| = m_1 \times m_2 = q^{n(n-1)} \prod_{i=1}^{n-1} (q^{2i} - 1) \times \frac{q^n + 1}{(2, q+1)}.$$

Moreover, we have $\frac{q^n+1}{(2,q+1)} = 2^p - 1$. We now consider two cases separately.

(*a*) $(2, q+1) = 1$. In this case, we get $q^n = 2(2^{p-1} - 1)$, an impossible.

(*b*) $(2, q+1) = 2$. In this case, we obtain $q^n - 1 = 4(2^{p-1} - 1)$, and since $(q^n - 1)^2$ divides $|P|$, it follows that $(2^{p-1} - 1)^2$ divides $|G|$, which is impossible by Lemma 3.2.

(2.2) If $P \cong C_r(2)$, then

$$|P| = |C_r(2)| = m_1 \times m_2 = 2^{r^2}(2^r + 1) \prod_{i=1}^{r-1} (2^{2i} - 1) \times (2^r - 1).$$

From $2^r - 1 = 2^p - 1$, it follows $r = p$. But then, we must have $2^{p^2} \big| |G|$, which is a contradiction.

(3) *The simple group P is isomorphic to none of the simple groups $A_{r-1}(q) \cong L_r(q)$, $(r, q) \neq (3, 2), (3, 4)$; $A_r(q) \cong L_{r+1}(q)$, $q - 1 | r + 1$; ${}^2A_{r-1}(q)$ and ${}^2A_r(q)$, $q + 1 | r + 1$, $(r, q) \neq (3, 3), (5, 2)$, where r is an odd prime.*

Since the proofs of all cases are similar, only the proofs for the simple groups $A_{r-1}(q)$, $(r, q) \neq (3, 2), (3, 4)$ and $A_r(q)$ with $q - 1 | r + 1$, are presented.

(3.1) If $P \cong A_{r-1}(q) \cong L_r(q)$, $(r, q) \neq (3, 2), (3, 4)$, then

$$|P| = m_1 \times m_2 = q^{\binom{r}{2}} \prod_{i=1}^{r-1} (q^i - 1) \times \frac{q^r - 1}{(r, q-1)(q-1)},$$

and

$$\frac{q^r - 1}{(r, q-1)(q-1)} = 2^p - 1.$$

Let $q = s^f$. If $s = 2$ and $f > 1$, then

$$2^{fr} - 1 \gneqq \frac{2^{fr} - 1}{(r, 2^f - 1)(2^f - 1)} = 2^p - 1.$$

Since $2^{fr}-1$ divides $|P|$, and so $|G|$, we get a contradiction by Theorem 2.2. In the case $s=2$ and $f=1$, we obtain that $r=p$, and so $P\cong A_{p-1}(2)\cong L_p(2)$, as required.

In the sequel we assume that s is an odd prime. First of all, we have

$$q^r-1 \gneqq \frac{q^r-1}{(r,q-1)(q-1)}=2^p-1,$$

or equivalently $q^r>2^p$. Since $s\in\pi(G)$, we assume that $s\in \mathrm{ppd}(2^k-1)$, for some k. Let $(2^k-1)_s=s^m$, where m is a natural number and

$$a:=(2^k-1)(2^{2k}-1)\cdots(2^{[\frac{p-1}{k}]k}-1),$$

(Note that k divides $p-1$, and so $[\frac{p-1}{k}]=\frac{p-1}{k}$). Then, by Lemma 2.2, we have

$$\begin{aligned} a_s &= \prod_{l=1}^{\frac{p-1}{k}}(2^{kl}-1)_s=\prod_{l=1}^{\frac{p-1}{k}} l_s(2^k-1)_s=\prod_{l=1}^{\frac{p-1}{k}} l_s s^m = s^{\frac{p-1}{k}m}\prod_{l=1}^{\frac{p-1}{k}} l_s \\ &= s^{\frac{p-1}{k}m}\left(\prod_{l=1}^{\frac{p-1}{k}} l\right)_s = s^{\frac{p-1}{k}m}((\tfrac{p-1}{k})!)_s = s^{\frac{p-1}{k}m}\cdot s^{\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]} = s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]}, \end{aligned}$$

and since $|G|_s=a_s$, it follows that

$$s^{f\frac{r(r-1)}{2}}=|P|_s\leqslant |G|_s=s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]}. \tag{2}$$

On the other hand, we have

$$\sum_{j=1}^{\infty}\left[\frac{p-1}{ks^j}\right]\leqslant\sum_{j=1}^{\infty}\frac{p-1}{ks^j}=\frac{p-1}{k}\sum_{j=1}^{\infty}\frac{1}{s^j}=\frac{p-1}{k}\cdot\frac{1}{s-1}\leqslant\frac{p-1}{k}.$$

If this is substituted in (2) and noting that $q^r>2^p$, then we obtain

$$\begin{aligned} s^{f\frac{r(r-1)}{2}} &\leqslant s^{\frac{p-1}{k}(m+1)}=\left(s^m\right)^{\frac{p-1}{k}}\cdot s^{\frac{p-1}{k}} \\ &< (2^k-1)^{\frac{p-1}{k}}\cdot(2^k-1)^{\frac{p-1}{k}}=(2^k-1)^{2\frac{p-1}{k}} \\ &< (2^k)^{2\frac{p-1}{k}}=2^{2(p-1)}<(2^p)^2<(q^r)^2=s^{2fr}, \end{aligned}$$

which implies that $\frac{r(r-1)}{2}<2r$, and so $r=3$. Thus $P\cong L_3(q)$, $q=s^f\neq 2,4$, and

$$\frac{q^3-1}{(3,q-1)(q-1)}=2^p-1. \tag{3}$$

Note that $|\mathrm{Out}(P)|=(3,q-1)\cdot f\cdot 2$, and hence

$$|\mathrm{Aut}(P)|=|P|\cdot|\mathrm{Out}(P)|=q^2(q^2-1)(q^3-1)\cdot f\cdot 2.$$

Moreover, subtracting 1 from both sides of Eq. (3) and easy computations show that

$$(q-1)(q+2)=\begin{cases} 4(2^{p-2}-1) & \text{if} \quad (3,q-1)=1, \\ 6(2^{p-1}-1) & \text{if} \quad (3,q-1)=3. \end{cases}$$

In what follows, we will consider two cases separately.

Case 1. $(3,q-1)=1$. Let $t\in \mathrm{ppd}(2^{p-2}-1)$ and $(2^{p-2}-1)_t=t^m$. Since $(q-1,q+2)=1$, we conclude that $(q-1)_t=t^m$ or $(q+2)_t=t^m$. If $(q-1)_t=t^m$, then since $(q-1)^2$ divides the order of P, it follows that $t^{2m}\big||P|$, and so $t^{2m}\big||G|$. But this contradicts Lemma 3.2, because $(2^{p-2}-1)^2$ does not divide the order of G. Therefore we may assume that $(q+2)_t=t^m$. Clearly $t\notin\pi(P)$, since

$$(q+2,\ q^2)=(q+2,\ q^2-1)=(q+2,\ q^3-1)=1.$$

On the other hand, since $t\in\mathrm{ppd}(2^{p-2}-1)$ and $t|2^{t-1}-1$, we deduce that $p-2|t-1$, and so $t\geqslant p-1$. In addition, from $(q^3-1)/(q-1)=2^p-1$, it follows that

$$q(q+1)=2(2^{p-1}-1),$$

and so $q=s^f|2^{p-1}-1$, since $(q,2)=1$. Thus $f\leqslant p-2<p-1\leqslant t$, which implies that $t\nmid f$. By what observed above we see that $t\notin\pi(\mathrm{Aut}(P))$, and so $t\in\pi(K)$. Assume now that $R\in \mathrm{Syl}_t(K)$. Certainly $R\in\mathrm{Syl}_t(G)$, and since K is nilpotent, $R\trianglelefteq G$. Now a (2^p-1)-Sylow subgroup of G, say T, acts fixed point freely on R by conjugation. This shows that the group RT is a Frobenius group with kernel R and complement T, and so

$$2^p-1\leqslant |R|-1\leqslant 2^{p-2}-1,$$

which is a contradiction.

Case 2. $(3,q-1)=3$. The proof goes in the same way as previous case.

(3.2) If $P\cong A_r(q)\cong L_{r+1}(q)$, $q-1|r+1$, then

$$|P|=m_1\times m_2=q^{\binom{r+1}{2}}(q^{r+1}-1)\prod_{i=2}^{r-1}(q^i-1)\times\frac{q^r-1}{q-1},$$

and

$$\frac{q^r-1}{q-1}=2^p-1.$$

Subtracting 1 from both sides of this equality, we obtain

$$q(q^{r-1}-1)=2(q-1)(2^{p-1}-1).$$

If q is even, then $q=2$ and $r=p$, which implies that $P\cong L_{p+1}(2)$, as required. Therefore, we may assume that $q=s^f$, where s is an odd prime and $f\geqslant 1$ a natural number. First of all, we have

$$q^r-1>\frac{q^r-1}{q-1}=2^p-1,$$

or equivalently $q^r > 2^p$. Since $s \in \pi(G)$, we assume that $s \in \mathrm{ppd}(2^k-1)$, for some k. Let $|2^k-1|_s = s^m$, where m is a natural number and

$$a := (2^k-1)(2^{2k}-1)\cdots(2^{[\frac{p-1}{k}]k}-1),$$

(Note that k divides $p-1$, and so $[\frac{p-1}{k}] = \frac{p-1}{k}$). Then, by Lemma 2.2, we have

$$\begin{aligned} a_s &= \prod_{l=1}^{\frac{p-1}{k}}(2^{kl}-1)_s = \prod_{l=1}^{\frac{p-1}{k}} l_s(2^k-1)_s = \prod_{l=1}^{\frac{p-1}{k}} l_s s^m = s^{\frac{p-1}{k}m}\prod_{l=1}^{\frac{p-1}{k}} l_s \\ &= s^{\frac{p-1}{k}m}\left(\prod_{l=1}^{\frac{p-1}{k}} l\right)_s = s^{\frac{p-1}{k}m}((\tfrac{p-1}{k})!)_s = s^{\frac{p-1}{k}m}\cdot s^{\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]} = s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]}, \end{aligned}$$

and since $|G|_s = a_s$, it follows that

$$s^{f\frac{r(r+1)}{2}} = |P|_s \leqslant |G|_s = s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]}. \tag{4}$$

On the other hand, we have

$$\sum_{j=1}^{\infty}\left[\frac{p-1}{ks^j}\right] \leqslant \sum_{j=1}^{\infty}\frac{p-1}{ks^j} = \frac{p-1}{k}\sum_{j=1}^{\infty}\frac{1}{s^j} = \frac{p-1}{k}\cdot\frac{1}{s-1} \leqslant \frac{p-1}{k}.$$

If this is substituted in (4) and noting that $q^r > 2^p$, then we obtain

$$\begin{aligned} s^{f\frac{r(r+1)}{2}} &\leqslant s^{\frac{p-1}{k}(m+1)} = (s^m)^{\frac{p-1}{k}}\cdot s^{\frac{p-1}{k}} \\ &< (2^k-1)^{\frac{p-1}{k}}\cdot(2^k-1)^{\frac{p-1}{k}} = (2^k-1)^{2\frac{p-1}{k}} \\ &< (2^k)^{2\frac{p-1}{k}} = 2^{2(p-1)} < (2^p)^2 < (q^r)^2 = s^{2fr}, \end{aligned}$$

which implies that $\frac{r(r+1)}{2} < 2r$, a contradiction.

(4) *The simple group P is isomorphic to none of the simple groups $E_6(q)$ and ${}^2E_6(q)$, $q > 2$.*

(4.1) If P is isomorphic to $E_6(q)$, $q = s^f$, then

$$|P| = |E_6(q)| = m_1\times m_2 = q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5-1)(q^3-1)(q^2-1)\times\frac{q^6+q^3+1}{(3,q-1)},$$

and

$$\frac{q^6+q^3+1}{(3,q-1)} = 2^p-1.$$

Thus, we have

$$q^9-1 > \frac{q^9-1}{q^3-1} = q^6+q^3+1 = (3,q-1)\cdot(2^p-1) \geqslant 2^p-1,$$

which yields that $q^9 > 2^p$. Again since $s \in \pi(G)$, we assume that $s \in \mathrm{ppd}(2^k-1)$, for some k. Suppose $|2^k-1|_s = s^m$, where m is a natural number and

$$a := (2^k-1)(2^{2k}-1)\cdots(2^{[\frac{p-1}{k}]k}-1).$$

Similarly to the previous case, we obtain

$$|G|_s = a_s = s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]},$$

and hence

$$\begin{aligned} s^{36\cdot f} = |P|_s &\leqslant |G|_s = s^{\frac{p-1}{k}m+\sum_{j=1}^{\infty}[\frac{p-1}{ks^j}]} \\ &\leqslant s^{\frac{p-1}{k}(m+1)} = (s^m)^{\frac{p-1}{k}} \cdot s^{\frac{p-1}{k}} \\ &< (2^k-1)^{\frac{p-1}{k}} \cdot (2^k-1)^{\frac{p-1}{k}} = (2^k-1)^{2\frac{p-1}{k}} \\ &< (2^k)^{2\frac{p-1}{k}} = 2^{2(p-1)} < (2^p)^2 < s^{18\cdot f}, \end{aligned}$$

which is a contradiction.

(4.2) The case when $P \cong {}^2E_6(q)$, $q > 2$, is similar to the previous case.

Case 3.2 $s(P) = 3$.

In this case we have $2^p - 1 \in \{m_2(P), m_3(P)\}$.

(1) $P \cong L_2(q)$, $4|q+1$. In this case $\frac{q-1}{2} = 2^p - 1$ or $q = 2^p - 1$. The first case is obviously impossible, since we obtain $q = 2^{p+1} - 1$ which must divides $|G|$. For the latter case, we first notice that q is a Mersenne prime and

$$|P| = |L_2(q)| = \frac{1}{(2, q-1)} q(q^2-1) = 2^p(2^{p-1}-1)(2^p-1).$$

Moreover, since $P \leqslant G/K \leqslant \mathrm{Aut}(P)$ and $|\mathrm{Aut}(P) : P| = 2$ we deduce that $2^{p-2}-1$ divides $|K|$. Let $r \in \mathrm{ppd}(2^{p-2}-1)$. Now we consider the Sylow r-subgroup R of K. Evidently $R \in \mathrm{Syl}_r(G)$ and $R \lhd G$ because K is a nilpotent subgroup. Now if $Q \in \mathrm{Syl}_q(G)$, then Q acts on R by conjugation and this action is fixed point free. Hence RQ is a Frobenius group with kernel R and complement Q, and we must have

$$q = 2^p - 1 \leqslant |R| - 1 \leqslant 2^{p-2} - 1,$$

which is a contradiction.

(2) $P \cong L_2(q)$, $4|q-1$. In this case we must have $q = 2^p - 1$ or $\frac{q+1}{2} = 2^p - 1$. The first case is obviously impossible, because $q - 1 = 2(2^{p-1}-1)$ and so $4 \nmid q-1$. If $\frac{q+1}{2} = 2^p - 1$, then $q = 2^{p+1} - 3$, and so

$$|P| = |L_2(q)| = \frac{1}{(2, q-1)} q(q^2-1) = 2^2(2^{p-1}-1)(2^p-1)(2^{p+1}-3).$$

Let $q = 2^{p+1} - 3 = s^f$, where s is a prime number. Evidently $s \geqslant 5$, and so

$$2^{p+1} \geqslant 2^{p+1} - 3 = s^f \geqslant 5^f \geqslant 2^{2f},$$

which forces $f \leqslant \frac{p+1}{2}$. Moreover, since

$$|\mathrm{Out}(P)| = |\mathrm{Aut}(P) : P| = (2, q-1) \cdot f = 2 \cdot f,$$

it follows that

$$|\mathrm{Aut}(P)| = 2^3(2^{p-1} - 1)(2^p - 1)(2^{p+1} - 3) \cdot f.$$

Let $r \in \mathrm{ppd}(2^{p-2} - 1) \subset \pi(G)$. Now, we claim that $(r, |\mathrm{Aut}(P)|) = 1$. Indeed, on the one hand, we have

$$\big(2^{p-2} - 1, \ \ 2^3(2^{p-1} - 1)(2^p - 1)(2^{p+1} - 3)\big) = 1,$$

whose validity is verified by direct computations. On the other hand, since $r|2^{r-1} - 1$, we deduce that $p - 2|r - 1$, and so $r \geqslant p - 1$. Combining this with the inequality $f \leqslant \frac{p+1}{2}$, we obtain

$$f \leqslant \frac{p+1}{2} < p - 1 \leqslant r,$$

which yields that $(r, f) = 1$. This completes the proof of our claim.

Therefore, from $(r, |\mathrm{Aut}(P)|) = 1$, it follows that $r \in \pi(K)$. As previous case, we consider the Sylow r-subgroup R of K, which is also the normal Sylow r-subgroup of G. Now a $(2^p - 1)$-Sylow subgroup of G, say Q, acts fixed point freely on R by conjugation. This shows that the group RQ is a Frobenius group with kernel R and complement Q, and so

$$2^p - 1 \leqslant |R| - 1 \leqslant 2^{p-2} - 1,$$

which is a contradiction.

(3) $P \cong L_2(q)$, $4|q$. Here, we must have $q - 1 = 2^p - 1$ or $q + 1 = 2^p - 1$. In the first case, we obtain $q + 1 = 2^p + 1\big||G|$, an impossible by Theorem 2.2. In the second case, we get $q = 2(2^{p-1} - 1)$, which is again a contradiction.

(4) $P \cong G_2(q)$, $3|q$. In this case $q^2 - q + 1 = 2^p - 1$ or $q^2 + q + 1 = 2^p - 1$. Now by easy calculate in both cases we obtain that $(2^{p-1} - 1)^2\big||G|$, which is a contradiction by Lemma 3.2.

(5) $P \cong {}^2G_2(q)$, $q = 3^{2n+1}$. In this case, we have

$$3^{2n+1} - 3^{n+1} + 1 = 2^p - 1 \quad \text{or} \quad 3^{2n+1} + 3^{n+1} + 1 = 2^p - 1.$$

Assume $3^{2n+1} - 3^{n+1} + 1 = 2^p - 1$. Now we easily deduce that

$$2(2^{\frac{p-1}{2}} - 1)(2^{\frac{p-1}{2}} + 1) = 3^{n+1}(3^n - 1).$$

If 3^{n+1} divides $2^{\frac{p-1}{2}} - 1$, then

$$3^n - 1 < 3^{n+1} \leqslant 2^{\frac{p-1}{2}} - 1 < 2^{\frac{p-1}{2}} + 1.$$

Hence, we obtain

$$3^{n+1}(3^n - 1) < 2(2^{\frac{p-1}{2}} - 1)(2^{\frac{p-1}{2}} + 1),$$

which is a contradiction. Assume now that 3^{n+1} divides $2^{\frac{p-1}{2}} + 1$. Then $2^{\frac{p-1}{2}} + 1 = k(3^{n+1})$, for some k, and so $3^{n+1} \leqslant 2^{\frac{p-1}{2}} + 1$. On the other hand, we observe that $2k(2^{\frac{p-1}{2}} - 1) = 3^n - 1$, and hence $3^n - 1 \geqslant 2(2^{\frac{p-1}{2}} - 1)$, i.e., $3^n \geqslant 2^{\frac{p+1}{2}} - 1$. Therefore, we have

$$2^{\frac{p+1}{2}} - 1 \leqslant 3^n < 3^{n+1} \leqslant 2^{\frac{p-1}{2}} + 1,$$

which is a contradiction. For other case the discussion is similar.

(6) $P \cong {}^2D_r(3)$, $r = 2^n + 1 \geqslant 3$. For this case, we have $\frac{3^r+1}{4} = 2^p - 1$ or $\frac{3^{r-1}+1}{2} = 2^p - 1$. In the first case, we obtain $2^2(2^p+1) = 3^2(3^{r-2}+1)$. Now, we consider a primitive prime divisor $r \in \mathrm{ppd}(2^{2p} - 1)$. Then $r \in \pi(2^p + 1) \subset \pi(P)$ and $r \notin \pi(G)$, which is a contradiction. In the second case, we get $2^{p+1} = 3(3^{r-2} + 1)$, which is a contradiction.

(7) $P \cong F_4(q)$, $2|q$. In this case we must have

$$q^4 + 1 = 2^p - 1 \quad \text{or} \quad q^4 - q^2 + 1 = 2^p - 1.$$

The first case obviously is impossible. In the latter case, we deduce

$$q^2(q^2 - 1) = 2(2^{p-1} - 1),$$

and so $(2^{p-1} - 1)^2$ divides $|G|$, which is a contradiction.

(8) $P \cong {}^2F_4(q)$, $q = 2^{2m+1} > 2$. Then

$$2^{2(2m+1)} - 2^{3m+2} + 2^{2m+1} - 2^{m+1} + 1 = 2^p - 1,$$

or

$$2^{2(2m+1)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1 = 2^p - 1.$$

Now, it is not difficult to see that any of equalities cannot hold.

(9) If $P \cong {}^2A_5(2)$ or $E_7(3)$, then $2^p - 1 = 7, 11, 757$ or 1093, which is a contradiction. If $P \cong E_7(2)$ then $2^p - 1 = 127$ and $p = 7$. But then we must have $43 \big| |G|$ which is a contradiction.

Case 3.3 $s(P) = 4, 5$.

In this case we have $2^p - 1 \in \{m_2(P), m_3(P), m_4(P), m_5(P)\}$.

(1) The cases $P \cong A_2(4)$, ${}^2E_6(2)$ are clearly impossible.

(2) If $P \cong {}^2B_2(2^{2m+1})$, $m \geqslant 1$, and $2^{2m+1} \pm 2^{m+1} + 1 = 2^p - 1$, then $m = 0$, against the fact $m \geq 1$. In the case when $2^{2m+1} - 1 = 2^p - 1$, it follows that $p = 2m + 1$ and we obtain $2^{2p} + 1 \big| |G|$, which is a contradiction.

(3) If $P \cong E_8(q)$, then $2^p - 1$ is one of the following:

(i) $q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$. This implies that

$$2(2^{p-1} - 1) = q(q-1)(q+1)(q^5 - q^4 + q^3 + 1),$$

which contradicts the fact that 8 divides $(q^2 - 1)$, if q is odd. If q is even, then $q = 2$ also gives a contradiction.

(ii) $q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$. This implies that

$$2(2^{p-1} - 1) = q(q-1)(q+1)(q^5 + q^4 + q^3 - 1),$$

which contradicts the fact that 8 divides $(q^2 - 1)$, if q is odd. If q is even, then $q = 2$ also gives a contradiction.

(iii) $q^8 - q^6 + q^4 - q^2 + 1$. This implies that

$$2(2^{p-1} - 1) = q^2(q-1)(q+1)(q^4 + q^2 - 1),$$

which contradicts the fact that 8 divides $(q^2 - 1)$, if q is odd. If q is even, then $q^2 = 2$ also gives a contradiction.

(iv) $q^8 - q^4 + 1$. This implies that $2(2^{p-1} - 1) = q^4(q^4 - 1)$, which also gives a contradiction.

The proof of this lemma is complete. □

Lemma 3.6 *G is isomorphic to* $\mathrm{Aut}(L)$.

Proof. By Lemma 3.5, P is isomorphic to L, and so $L \leqslant G/K \leqslant \mathrm{Aut}(L)$. Since $|\mathrm{Out}(L))| = 2$, $G/K \cong L$ or $G/K \cong \mathrm{Aut}(L)$. In the first case, $|K| = 2$ and so $K \leqslant Z(G)$ which forces G possesses an element of order $2 \cdot (2^p - 1)$, a contradiction. In the later case, one can easily deduce that $K = 1$ and $G \cong \mathrm{Aut}(L)$, as required. □

The proof of the theorem is complete. □

4 Appendix

In a series of papers, it was shown that many finite simple groups are OD-characterizable or 2-fold OD-characterizable. Table 4 lists finite simple groups which are currently known to be k-fold OD-characterizable for $k \in \{1, 2\}$.

Table 4. Some non-abelian simple groups S with $h_{\mathrm{OD}}(S) = 1$ or 2.

S	Conditions on S	h_{OD}	Refs.
$\mathbb{A}_n$	$n = p, p+1, p+2$ (p a prime)	1	[23], [26]
	$5 \leqslant n \leqslant 100, n \neq 10$	1	[12], [15], [22], [24], [43]
	$n = 106,\ 112$	1	[33]
	$n = 10$	2	[25]
$L_2(q)$	$q \neq 2, 3$	1	[23], [26], [41]
$L_3(q)$	$\lvert\pi(\frac{q^2+q+1}{d})\rvert = 1,\ d = (3, q-1)$	1	[26]
$U_3(q)$	$\lvert\pi(\frac{q^2-q+1}{d})\rvert = 1,\ d = (3, q+1), q > 5$	1	[26]
$L_4(q)$	$q \leqslant 17$	1	[1, 3]
$L_3(9)$		1	[42]
$U_3(5)$		1	[40]
$U_4(7)$		1	[3]
$L_n(2)$	$n = p$ or $p+1$, for which $2^p - 1$ is a prime	1	[3]
$L_9(2)$		1	[14]
$R(q)$	$\lvert\pi(q \pm \sqrt{3q} + 1)\rvert = 1,\ q = 3^{2m+1},\ m \geqslant 1$	1	[26]
$\mathrm{Sz}(q)$	$q = 2^{2n+1} \geqslant 8$	1	[23], [26]
$B_m(q), C_m(q)$	$m = 2^f \geqslant 4,\ \lvert\pi\big((q^m+1)/2\big)\rvert = 1,$	2	[2]
$B_2(q) \cong C_2(q)$	$\lvert\pi\big((q^2+1)/2\big)\rvert = 1,\ q \neq 3$	1	[2]
$B_m(q) \cong C_m(q)$	$m = 2^f \geqslant 2,\ 2 \mid q,\ \lvert\pi\big(q^m+1\big)\rvert = 1,\ (m, q) \neq (2, 2)$	1	[2]
$B_p(3), C_p(3)$	$\lvert\pi\big((3^p-1)/2\big)\rvert = 1,\ p$ is an odd prime	2	[2], [26]
$B_3(5), C_3(5)$		2	[2]
$C_3(4)$		1	[21]
S	A sporadic simple group	1	[26]
S	A simple group with $\lvert\pi(S)\rvert = 4,\quad S \neq \mathbb{A}_{10}$	1	[39]
S	A simple group with $\lvert S\rvert \leqslant 10^8,\quad S \neq \mathbb{A}_{10},\ U_4(2)$	1	[37]
S	A simple $C_{2,2}$- group	1	[23]

Although we have not found a simple group which is k-fold OD-characterizable for $k \geqslant 3$, but among non-simple groups, there are many groups which are k-fold OD-characterizable for $k \geqslant 3$. As an easy example, if P is a p-group of order p^n, then $h_{\mathrm{OD}}(P) = \nu(p^n)$. In connection with such groups, Table 5 lists finite non-solvable groups which are currently known to be OD-characterizable or k-fold OD-characterizable with $k \geqslant 2$.

In Table 4, q is a power of a prime number.

Table 5. Some non-solvable groups G with certain $h_{\rm OD}(G)$.

G	Conditions on G	$h_{\rm OD}(G)$	Refs.
$\mathrm{Aut}(M)$	M is a sporadic group $\neq J_2, M^cL$	1	[23]
$\mathbb{S}_n$	$n=p,\ p+1$ ($p \geqslant 5$ is a prime)	1	[23]
M	$M \in \mathcal{C}_1$	2	[25]
M	$M \in \mathcal{C}_2$	2	[26]
M	$M \in \mathcal{C}_3$	8	[25]
M	$M \in \mathcal{C}_4$	3	[12, 15, 22, 24, 33]
M	$M \in \mathcal{C}_5$	2	[25]
M	$M \in \mathcal{C}_6$	3	[25]
M	$M \in \mathcal{C}_7$	6	[22]
M	$M \in \mathcal{C}_8$	1	[38]
M	$M \in \mathcal{C}_9$	9	[38]
M	$M \in \mathcal{C}_{10}$	1	[40]
M	$M \in \mathcal{C}_{11}$	3	[40]
M	$M \in \mathcal{C}_{12}$	6	[40]
M	$M \in \mathcal{C}_{13}$	1	[34]

$$
\begin{aligned}
\mathcal{C}_1 &= \{\mathbb{A}_{10}, J_2 \times \mathbb{Z}_3\}\\
\mathcal{C}_2 &= \{S_6(3), O_7(3)\}\\
\mathcal{C}_3 &= \{\mathbb{S}_{10},\ \mathbb{Z}_2 \times \mathbb{A}_{10},\ \mathbb{Z}_2 \cdot \mathbb{A}_{10},\ \mathbb{Z}_6 \times J_2,\ \mathbb{S}_3 \times J_2,\ \mathbb{Z}_3 \times (\mathbb{Z}_2 \cdot J_2),\\
&\qquad (\mathbb{Z}_3 \times J_2) \cdot \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathrm{Aut}(J_2)\}.\\
\mathcal{C}_4 &= \{\mathbb{S}_n,\ \mathbb{Z}_2 \cdot \mathbb{A}_n,\ \mathbb{Z}_2 \times \mathbb{A}_n\},\ \text{ where } 9 \leqslant n \leqslant 100 \text{ with } n \neq 10, p, p+1 \ (p \text{ a prime})\\
&\qquad \text{or } n = 106,\ 112.\\
\mathcal{C}_5 &= \{\mathrm{Aut}(M^cL),\ \mathbb{Z}_2 \times M^cL\}.\\
\mathcal{C}_6 &= \{\mathrm{Aut}(J_2),\ \mathbb{Z}_2 \times J_2,\ \mathbb{Z}_2 \cdot J_2\}.\\
\mathcal{C}_7 &= \{\mathrm{Aut}(S_6(3)),\ \mathbb{Z}_2 \times S_6(3),\ \mathbb{Z}_2 \cdot S_6(3),\ \mathbb{Z}_2 \times O_7(3),\ \mathbb{Z}_2 \cdot O_7(3),\ \ \mathrm{Aut}(O_7(3))\}.\\
\mathcal{C}_8 &= \{L_2(49) : 2_1,\ L_2(49) : 2_2,\ L_2(49) : 2_3\}.\\
\mathcal{C}_9 &= \{L \cdot 2^2,\ \mathbb{Z}_2 \times (L : 2_1),\ \mathbb{Z}_2 \times (L : 2_2),\ \mathbb{Z}_2 \times (L \cdot 2_3),\ \mathbb{Z}_2 \cdot (L : 2_1),\\
&\qquad \mathbb{Z}_2 \cdot (L : 2_2),\ \mathbb{Z}_2 \cdot (L \cdot 2_3),\ \mathbb{Z}_4 \times L,\ (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L\},\ \text{ where } L = L_2(49).\\
\mathcal{C}_{10} &= \{U_3(5),\ U_3(5) : 2\}\\
\mathcal{C}_{11} &= \{U_3(5) : 3,\ \mathbb{Z}_3 \times U_3(5),\ \mathbb{Z}_3 \cdot U_3(5)\}\\
\mathcal{C}_{12} &= \{L : \mathbb{S}_3,\ \mathbb{Z}_2 \cdot (L : 3),\ \mathbb{Z}_3 \times (L : 2),\ \mathbb{Z}_3 \cdot (L : 2),\ (\mathbb{Z}_2 \times L) \cdot \mathbb{Z}_2,\\
&\qquad (\mathbb{Z}_3 \cdot L) \cdot \mathbb{Z}_2\}, \text{ where } L = U_3(5).\\
\mathcal{C}_{13} &= \{\mathrm{Aut}(O_{10}^{+}(2), \mathrm{Aut}(O_{10}^{-}(2)\},
\end{aligned}
$$

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